9.1 Stability and the phase plane

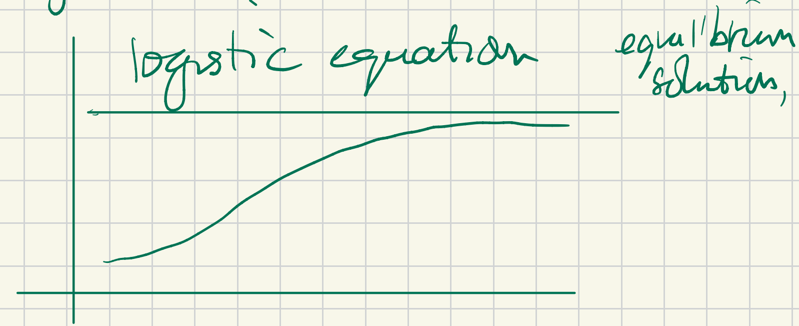
Vocabulary we already know:

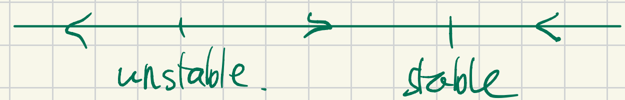
- phase plane, direction field, slope field
- phase portrait of a 2-dimensional autonomous system = the equation doesn't change with time.
- solution curve = trajectory
- critical point, equilibrium solution

derivatives are 0.

Vocabulary that is new:

- node (proper, improper)
- sink source
- spiral sink = stable spiral point
- spiral source = unstable spiral point
- stable center
- saddle point
- stable, asymptotically stable
- unstable



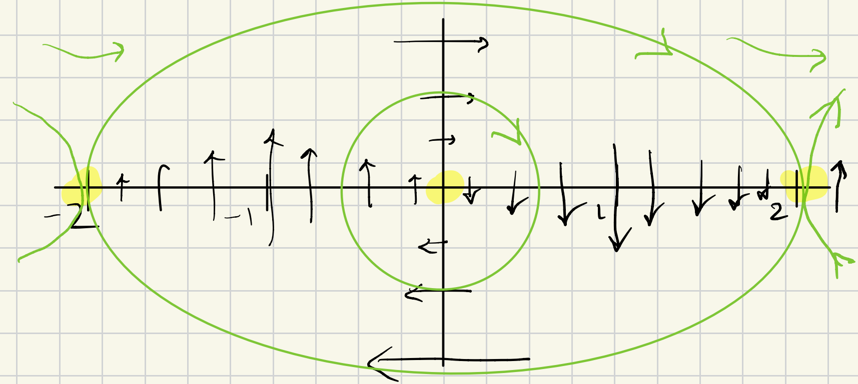


Page 512 question 9 (differently worded, plus a bit, on the HW but not to be handed in)

In the equation $x'' + 4x - x^3 = 0$ put
$x' = y$, $y' = x^3 - 4x$
and use a computer system or graphing calculator to construct a phase portrait and direction field. Find the critical points and classify them as sink, source, saddle point, spiral point, stable center, going by what they look like.

Solution. Draw picture by plotting vectors
e.g. when $y = 0$, $(x', y') = (y, x^3 - 4x)$

The critical points occur when
$(x', y') = (0, 0)$ i.e. $(y, x^3 - 4x) = (0, 0)$
$x^3 - 4x = x(x-2)(x+2)$.
Critical points $(0,0)$, $(-2,0)$, $(2,0)$



We get a stable center at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, saddle points at $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Pre-class Warm-up!!!

- The system of equations $x' = Ax$ where

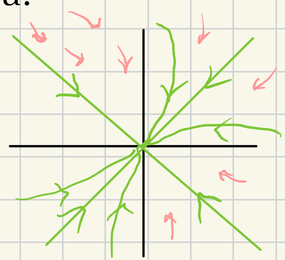
$A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$ that is, $\begin{aligned} x' &= -x + 3y \\ y' &= 3x - y \end{aligned}$

has general solution $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

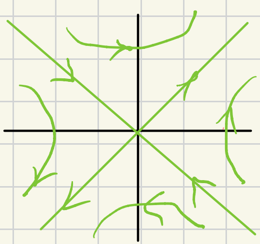
(because A has eigenvalues $\lambda = 2, -4$
and eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$)

What is the correct phase portrait for this system?

a.



✓ b.



Another question: is it easier to draw the phase portrait knowing

a. the original equations, or

b. the solutions to the system.

Yet another question: what is the difference between a phase portrait and a direction field?

Is there a difference

a. Yes

b. No

Types of critical point. Critical points may be

Stable Sufficiently close to the critical point, trajectories remain close to the critical point.

Unstable = not stable.

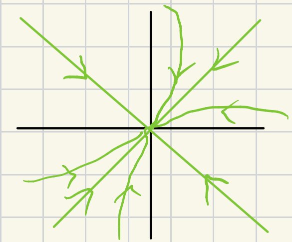
Asymptotically stable (implies stable)
Stable, and every trajectory goes in to the critical point.

Examples of the form $x' = Ax$.
The critical point is at the origin.

1. $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $\lambda = -1, -3$, e-vecs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

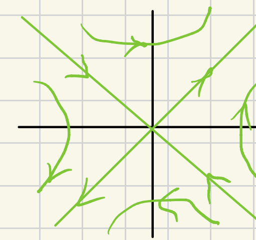
General solution
$c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This is stable, asymptotically stable



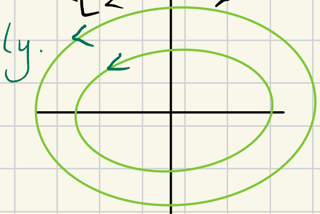
2. $A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$ $\lambda = 2, -4$
e-vecs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This is unstable



3. $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}$, $x = c_1\begin{bmatrix} \cos 2t \\ \frac{1}{2}\sin 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t \\ -\frac{1}{2}\cos 2t \end{bmatrix}$

stable but not asymptotically.



Question: do the trajectories go
A clockwise?
B counterclockwise? ✓

Separately from whether a critical point is stable or unstable, it can be a

- node, which can be proper or improper,
- a saddle point,
- a spiral point, or
- a center

A node is a critical point so that

- either every trajectory approaches it or every trajectory recedes from it, and
- every trajectory is tangent to some straight line through the critical point.

The node is proper if all the tangent lines are distinct.

A node or a spiral point can be a source or a sink, depending on its stability.

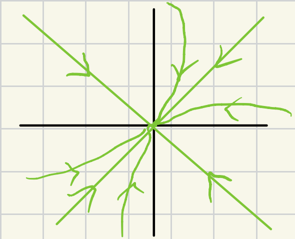
Examples from the last page: $x' = Ax$ with a critical point at the origin.

1. $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $\lambda = -1, -3$, e-vecs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

General solution

$c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

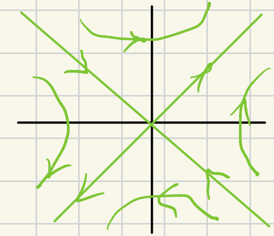
proper node



2. $A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$ $\lambda = 2, -4$

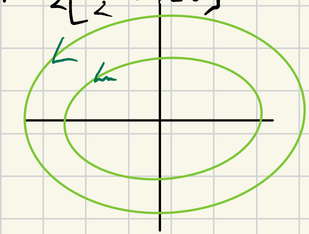
e-vecs $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

not a node



3. $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}$, $x = c_1\begin{bmatrix} \cos 2t \\ \frac{1}{2}\sin 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t \\ -\frac{1}{2}\cos 2t \end{bmatrix}$

not a node



Question: for each of 1, 2, 3, is it a node, or not a node?

Example: $x' = Ax$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Eigenvalues: $\lambda = 1, 1$ only one eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e^{At}\underline{c}$ solves $x' = Ax$

where

$$e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

Independent solutions: $\begin{bmatrix} e^t \\ 0 \end{bmatrix}, \begin{bmatrix} te^t \\ e^t \end{bmatrix}$

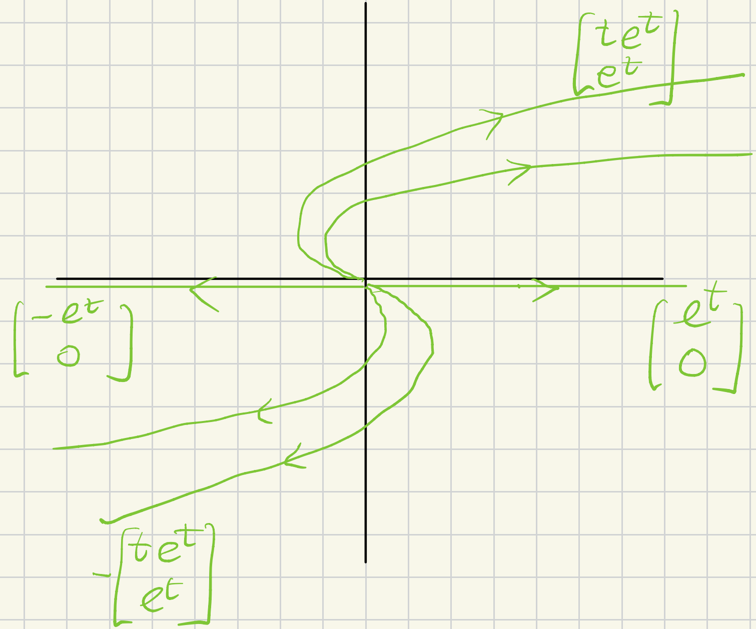
We can check that every trajectory is tangent to the x-axis as it approaches the origin.

Take a solution $\underline{x} = c_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} te^t \\ e^t \end{bmatrix}$ so

$\underline{x}' = \begin{bmatrix} c_1 e^t + c_2 e^t + c_2 te^t \\ c_2 e^t \end{bmatrix}$ and its slope is

$$\frac{c_2 e^t}{c_1 e^t + c_2 e^t + c_2 te^t} = \frac{c_2}{c_1 + c_2 + c_2 t} \to 0 \text{ as } t \to -\infty$$

Note: the curve gets close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to -\infty$



This is an improper node: every trajectory is tangent to the x-axis.

Question: Is it

a. stable b. unstable ✓

Spiral points

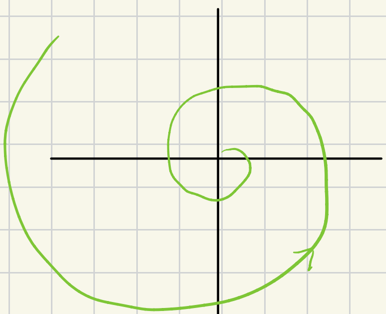
A critical point where the trajectories wind round and round, and either approach the critical point or leave it, is called a spiral.

If the trajectories wind round and round and are closed, the critical point is called a center.

$x' = Ax$

1. $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $\lambda = -1+i$, $-1-i$ e-vec: $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\begin{bmatrix} 1 \\ i \end{bmatrix}$

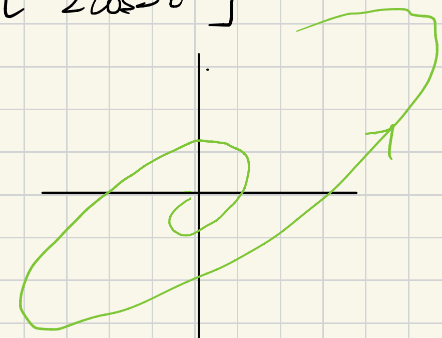
Solutions: $e^{-t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ $e^{-t}\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$



2. $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ $\lambda = 2 \pm 3i$

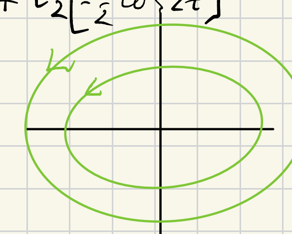
Solutions: $e^{2t}\begin{bmatrix} 3\cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix}$, $e^{2t}\begin{bmatrix} 3\cos 3t + \sin 3t \\ 2\sin 3t \end{bmatrix}$

The cos and sin terms give a spiral or a center. Because of the factor e^{2t}, it is spiral source, it is unstable.



3. $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}$, $x = c_1\begin{bmatrix} \cos 2t \\ \frac{1}{2}\sin 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t \\ -\frac{1}{2}\cos 2t \end{bmatrix}$

Is the critical point
a. stable?
b. unstable?
c. asymptotically stable?



This is a center

Typical question, like questions 13-20:
The system of equations given below has a critical point when (x,y) = (0,0). Classify this critical point, determining whether it is

- stable, asymptotically stable or unstable, and
- a proper node, an improper node, a spiral point, a saddle point or a stable center.

	a	b	c	d	e	f
Stable				✓		
As. stable				✓		
Unstable				x		
Proper node				x		
Improper node				x		
Source				x		
Sink				✓		
Spiral point				✓		
Saddle point				x		
Stable center				x